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PLATES AND TANKS ON ELASTIC FOUNDATIONS—AN APPLICATION OF FINITE ELEMENT METHOD*

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Abstract—In this paper, the problems of slabs and tanks (either isotropic or orthotropic) resting either on a semi-infinite elastic continuum or on individual springs (of the so-called Winkler's type) are solved by the finite element method. Re-entrant corners, rigid walls on the slabs, concentrated moments due to bending of columns, etc., involve little computational difficulty in the method presented.

1. INTRODUCTION

A DETAILED description of the finite element method for the analysis of elastic isotropic and orthotropic slabs can be found in a previous paper by the authors [1]. The slab is first divided into a number of small elements which are then joined at a discrete number of nodal points where continuity and equilibrium conditions are established. From the resulting equations the deformations can be determined. In the present case, the stiffness coefficients of the foundation at the various nodes are simply added to those of the plate elements. After the nodal displacements have been determined, the contact pressures and the plate moments can be worked out easily by simple matrix operations.

In the problems of beams or plates resting on an elastic foundation different assumptions have been introduced to simplify the mathematical formulation. The first of these is that no separation occurs when negative reactions are present. This assumption is in practice a reasonable one as usually the weight of the structure imposes an initial pre-compression.

The second assumption frequently made is less tenable. This assumes that no interaction exists between adjacent points of the foundation and that this reacts as a series of isolated springs. This so-called Winkler's foundation is obviously a fiction when true subgrades are considered. Many authors have tried to justify this for special cases but the errors involved are of a serious nature. It will be shown that, in the proposed approach, little additional difficulty is experienced in treating the foundation as a continuum.

The problem of beams on elastic foundation both of continuum and of Winkler type have been extensively investigated by Hetenyi [2], Zemochkin [3], Chai [4] and many others, while that of plates and tanks on elastic foundation has received scant attention. Holl [5] solved the problem of an infinite plate on an elastic half-space under axi-symmetrical loadings, while Naghdi and Rowley [6], Pickett and McCormick [7], and Frederick [8] tackled finite and infinite plates on Winkler's type of foundation. Vint and Elgood [9] dealt with a finite rectangular plate on Winkler's foundation by the Raleigh–Ritz method, while Allen and Severn [10] solved the same problem, but with more

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complex boundary conditions, by relaxation of two second order differential equations. Recently James [11] solved a finite rectangular plate on elastic half-space by finite differences.

2. FOUNDATION STIFFNESS MATRICES

(a) *Winkler's foundation*

In Winkler's foundation, the contact pressure p is regarded as being proportional to the deflection w, or simply

$$p = kw \tag{1}$$

where k is the modulus of subgrade reaction.

Such property is in fact exhibited by a foundation of a heavy liquid type, or by a foundation consisting of independent springs.

For a division into rectangular finite element mesh (Fig. 1) with sides a and b, equation (1) can be rewritten as

$$P_i = \alpha_i abk_i w_i \tag{2}$$

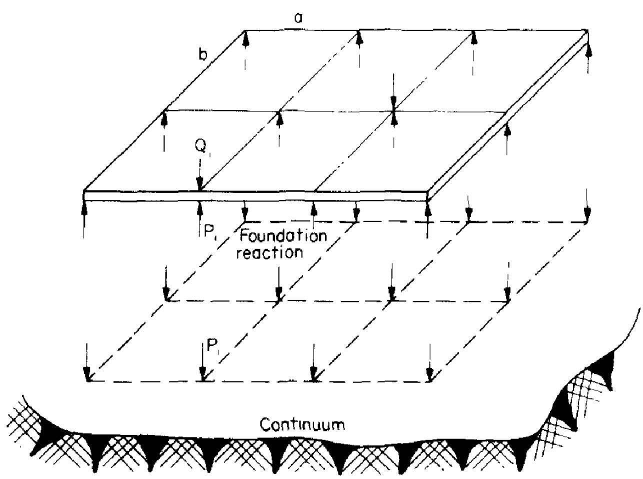

FIG. 1. A plate and its foundation.

where P_i is the normal force at node i and α_i is a coefficient which takes the value of $\frac{1}{4}$ at corners, $\frac{1}{2}$ at the sides, 1 at interior nodes and $\frac{3}{4}$ at right angled re-entrant corners to account for the area contributing to the nodal forces.

In matrix form, this can be written as

$$[P] = abk[\alpha]\{w\} \tag{3}$$

in which α is a purely diagonal matrix.

(b) *Isotropic elastic half-space*

In this treatment which now allows for an interaction of the various parts of the foundation it is convenient to assume a constant pressure acting on the rectangle $a \times b$

around each nodal point. This pressure will have now a magnitude of P_i/ab, and will vary from node to node.

The deflection of any point n due to a point load at i on an isotropic elastic half-sapce is given by the Boussinesq equation

$$w_{ni} = \frac{F(1 - \nu_0^2)}{\pi E_0 r_n} \tag{4}$$

where ν_0 is the Poisson's ratio of the foundation, E_0 is the Young's modulus of the foundation and r_n is the radial distance between points i and n.

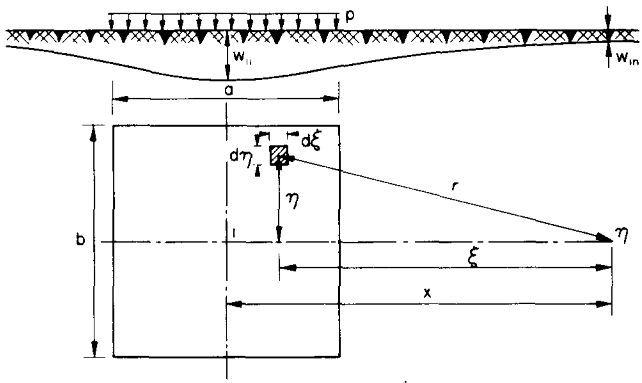

FIG. 2. Vertical displacements due to uniformly loaded rectangular area on isotropic half-space.

The deflection at the centre of the uniformly loaded rectangular area $a \times b$ can be obtained by integrating equation (4) over the rectangular area (Fig. 2)

$$w_{ii} = 2\int_{\xi=0}^{\xi=a/2} 2\int_{\eta=0}^{\eta=b/2} \frac{P_i(1-\nu_0^2)}{ab\pi E_0} \frac{d\xi\, d\eta}{\sqrt{(\xi^2+\eta^2)}} = \frac{P_i(1-\nu_0^2)}{a\pi E_0} f_{ii}. \tag{5}$$

Some values of f_{ii} are given in Table 1 for various ratios of b/a.

TABLE 1

b/a	2/3	1	2	3	4	5
f_{ii}	4·265	3·525	2·406	1·867	1·543	1·322

For a point outside the loaded area, the same integration can be done, but a good enough approximation can be achieved by using equation (4), where F is now the total load over the rectangle, i.e. P_i, and r_n the centre to centre distance. Some exact results for f_{ii} are given in Table 2 and are compared with the approximate values. It can be seen that even for $x = a$, the error is only some 4 per cent, and that it decreases rapidly with increase of x.

Therefore, for any set of grid points, the deflections can be written as

$$\{w\} = \frac{(1-\nu_0^2)}{\pi E_0 a}[f_f]\{P\} \tag{6}$$

TABLE 2

x/a	1	2	3	4	5	6	7	8	9	10
Exact	1·038	0·505	0·333	0·251	0·200	0·167	0·143	0·125	0·111	0·100
Approx.	1·000	0·500	0·333	0·250	0·200	0·167	0·143	0·125	0·111	0·100

where $[f_f]$ is the flexibility matrix of the foundation, obtained for off-diagonal terms by (4) and for diagonal terms by (5). Inverting one can write

$$\{P\} = \frac{\pi E_0 a}{(1-\nu_0^2)}[K_f]\{w\} \tag{7}$$

where $[K_f] = [f_f]^{-1}$.

3. THE COMPLETE STIFFNESS FORMULATION

This matrix has now to be combined with that of a plate subdivided into finite elements. In reference (1) such a matrix is given in equation (8) connecting the nodal forces N and displacements U

$$\{N\} = [S]\{U\}. \tag{8}$$

For each nodal force N_i and displacement $\{U_i\}$ three components are present. These correspond to lateral displacement w_i and two rotations θ_{xi} and θ_{yi}. As no angular continuity is assumed between the foundation and the plate it is possible by partial inversion to eliminate the rotations and corresponding moments from the above relation. Noting that if Q_i represents an external applied load to a node $Q_i - P_i$ is the effective external force acting on that node and we can write, for an isotropic plate

$$\{Q\} - \{P\} = \frac{D}{15ab}[K_p]\{w\} \tag{9}$$

in which D is the plate rigidity equal to

$$\frac{E_p t^3}{12(1-\nu_p^2)}.$$

Thus for example, for a foundation of the Winkler type we have on eliminating P by the use of equation (3)

$$\{Q\} = \frac{D}{15ab}\left([K_p] + \frac{15ab}{D}abk[\alpha]\right)\{w\} \tag{10}$$

$$= \frac{D}{15ab}([K_p] + \bar{k}[\alpha])\{w\}.$$

Similarly for an isotropic half-space

$$\{Q\} = \frac{D}{15ab}\left([K_p] + \frac{15ab}{D}\frac{\pi E_0 a}{(1-\nu_0^2)}[K_f]\right)\{w\} \tag{11}$$

$$= \frac{D}{15ab}([K_p] + \gamma[K_f])\{w\}$$

where

$$\gamma = \frac{180\pi a^2 b}{t^3} \frac{(1-\nu_p^2)}{(1-\nu_0^2)} \frac{E_0}{E_p}.$$

4. SOLUTION

A direct solution of equations (10) and (11) gives the deflections, which in turn will give bending moments of the slab and the contact pressures.

However, if only the contact pressure is required, as often is the case, we can simplify the procedure. For the case of elastic half-space we have from (9) and (6)

$$\{Q\}-\{P\} = \frac{D}{15ab}[K_p]\{w\}$$

$$= \frac{D(1-\nu_0^2)}{15\pi a^2 bE_0}[K_p][f_f]\{P\}$$

or

$$\{Q\} = \left(\frac{D(1-\nu_0^2)}{15\pi a^2 bE_0}[K_p][f_f]+[I]\right)\{P\} \tag{14}$$

where $[I]$ is the unit matrix.

The solution of equation (14) yields $[P]$, and the contact pressures can be obtained by dividing by the appropriate areas. This approach saves the inversion of the $[f_f]$ matrix.

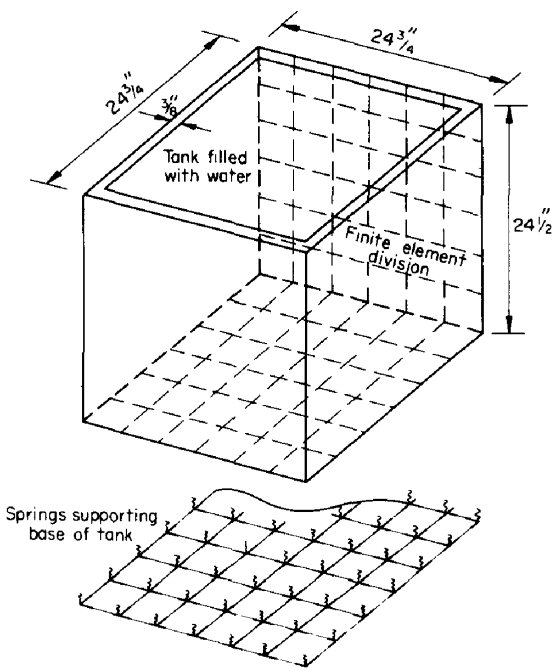

FIG. 3. Tank on elastic foundation.

5. TANKS ON AN ELASTIC FOUNDATION

The finite element method described can be extended to deal with such problems as tanks on elastic foundations, Fig. 3. Here the horizontal elements of the base slab are coupled with vertical ones at the side walls by appropriately re-orientating the forces at the nodes where the junction occurs.

Clearly 'in plane' forces now arise both in the wall and base slabs and in principle it would appear necessary to include their effect on deformation.

It is, however, well known that in such structures the major part of deformation is due to bending and it is customary to neglect the 'in plane' extensions. This introduces certain special features required in the solution. In particular if no 'in plane' deformation occurs the vertical deformations of the foundation slab must vary linearly along its edge.

Similar conditions will pertain along the vertical sides.

Such additional relations are necessary to eliminate certain unknowns from that system of displacement equations which can then be solved in the normal manner.

6. NUMERICAL EXAMPLES

(a) *Uniformly loaded square plate on isotropic half-space*

A series of computations were carried out, and the average contact pressures at several chosen points are plotted against the logarithms of the relative rigidity $\gamma = 180\pi(E_0/E_p)(a/t)^3$*, as shown in Figs. 4 and 5. As is expected, the contact pressures

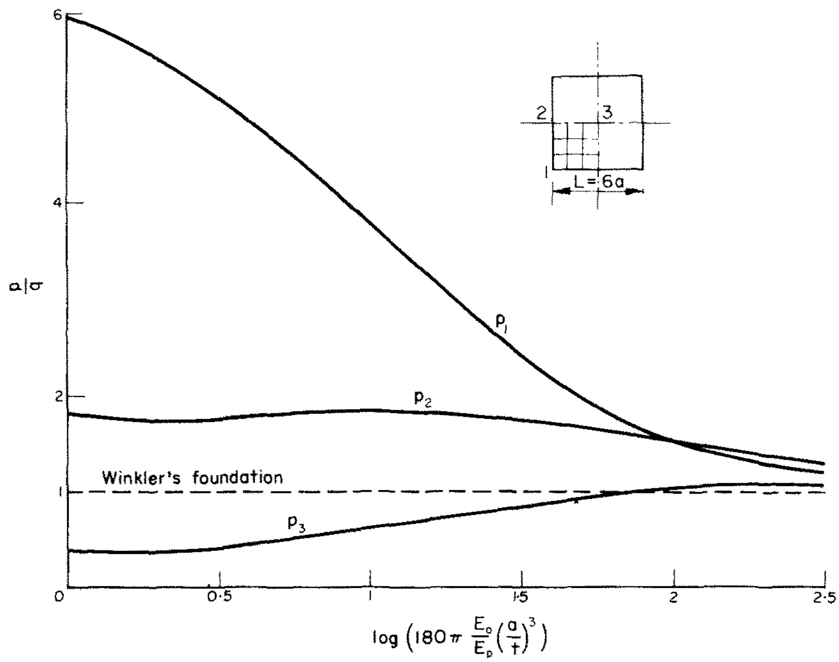

Fig. 4. Contact pressures of uniformly loaded square plate on isotropic half-space at points 1, 2 and 3 for various values of γ.

* In this case $1-\nu_p^2/1-\nu_0^2$ is taken as unity.

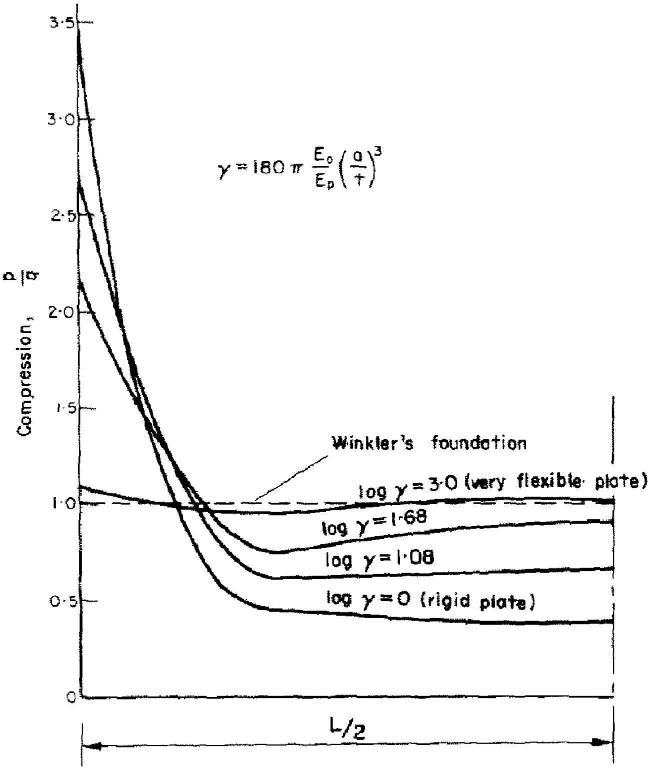

FIG. 5. Contact pressure variation along centre line of plate under uniform load q.

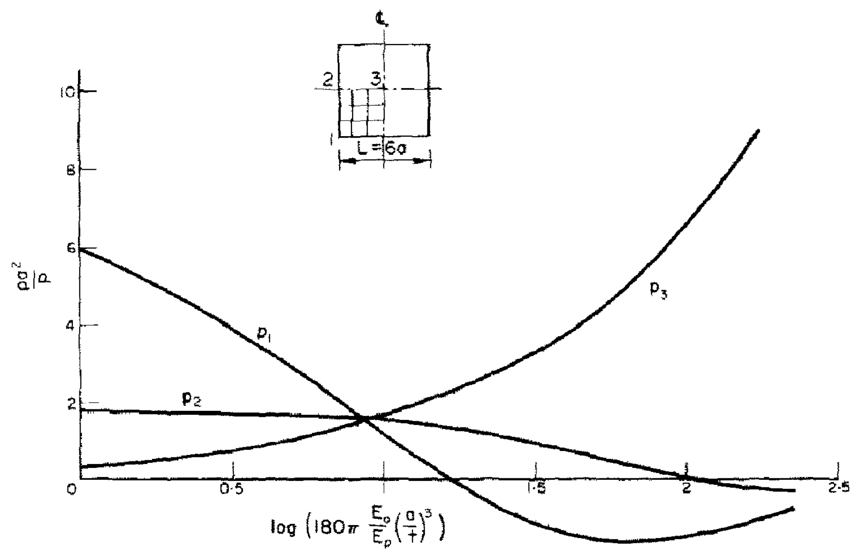

FIG. 6. Contact pressures of square plate on isotropic half-space with concentrated load at centre (P) at points 1, 2 and 3 for various values of γ.

approach a very high value at the corner (infinity for an exact solution) for the case of a rigid plate, and they all converge to the same value, equal to that of the uniform load, when the plate is very flexible.

This simple example illustrates the danger of using indiscriminately the Winkler approximation which now gives the trivial answer of uniform load whatever the plate stiffness.

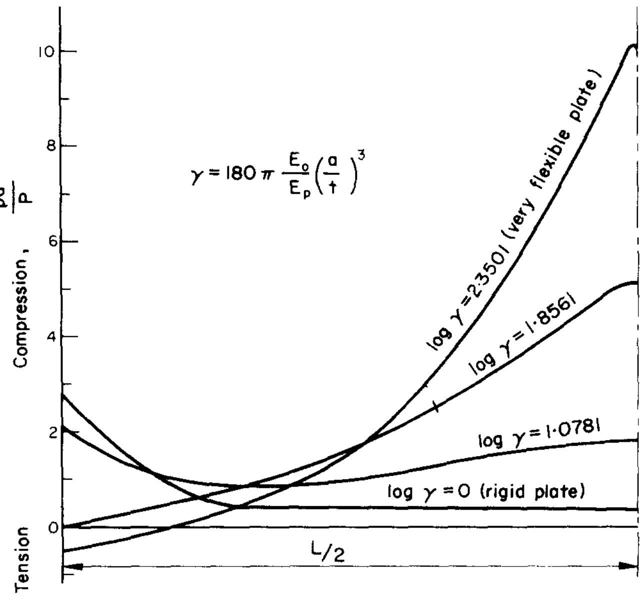

FIG. 7. Contact pressure variation along centre line of plate on isotropic half-space under concentrated load P.

(b) *Square plate on isotropic half-space with a concentrated load at centre* (*Figs.* 6 *and* 7)

The average contact pressures are again plotted against the relative rigidities $\gamma = 180\pi(E_0/E_p)(a/t)^3$ and the results are again not unexpected. For the case of a very flexible plate, the contact pressure at the centre reaches a very high value, as the plate now offers very little help in spreading out the load.

For comparison the contact pressure variation along the centre line of the same plate but resting on a Winkler type foundation is shown in Fig. 8.

Many authors have endeavoured to obtain equivalence between the Winkler type constants and the moduli of the foundation continuum. The temptation to do so here is resisted as it is clear that such comparative figures can only be valid for very limited ranges of plate size and loading.

(c) *Deformation of a plate with four loads and supported in springs* (*Fig.* 9)

This simple case appeared to be one of the few cases listed in the literature for which actual solutions have been computed and tested on experiment and therefore, was chosen for comparative purposes.

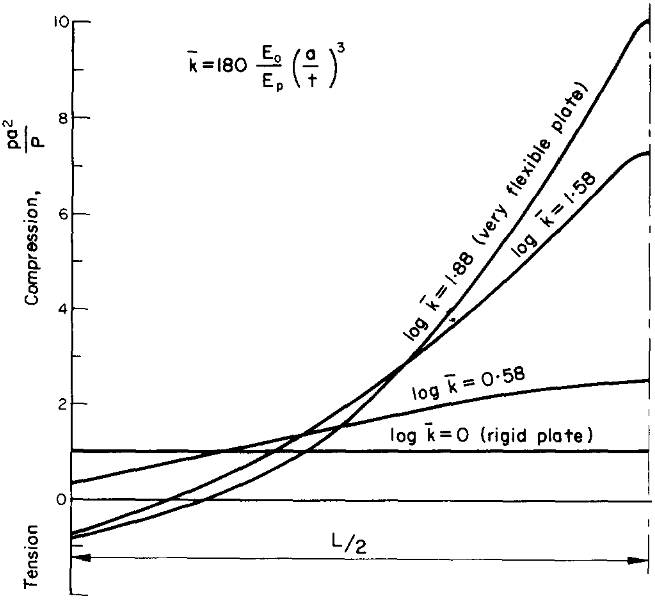

FIG. 8. Contact pressure variation along centre line of plate on springs under concentrated load P.

The deflections are worked out by the finite element method and compared with those presented by Vint and Elgood [9] in Table 3. The agreement is excellent even though the mesh used is only 6 × 6 for the whole plate. It is worth noting that here the actual loading locations fall between the nodes and the loads have to be distributed to the surrounding nodes by suitable static apportioning.

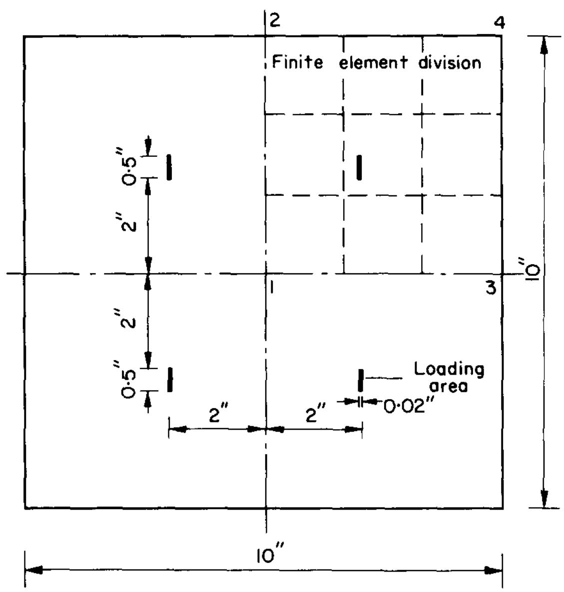

FIG. 9. Deformation of a plate with four loads and supported on springs.

TABLE 3

	w_1	w_2	w_3	w_4
Raleigh–Ritz [9]	0·1457	0·1338	0·1283	0·1175
Experimental [9]	0·1440	0·1365	0·1270	0·1213
Finite element 6 × 6	0·1465	0·1359	0·1277	0·1183
Multiplier	inches			

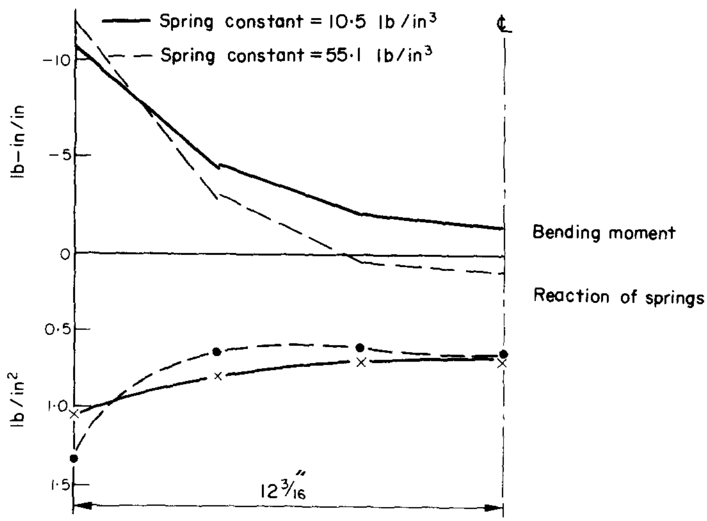

FIG. 10. Bending moment of base slab and spring reaction along centre line.

TABLE 4

Spring constant	10·5 lb/in.³				19·5 lb/in.³			
	w_1	w_2	M_{x1}	M_{x2}	w_1	w_2	M_{x1}	M_{x2}
Experimental			0·322	−10·55			0·555	−10·60
Kantorovich	104	42	−1·349	−10·62	63	35	−0·593	−11·19
Finite element 6 × 6	99	65	−1·37	−10·55	58	33	−0·32	−11·13
Multiplier	10^{-3} in.		lb in./in.		10^{-3} in.		lb in./in.	

Spring constant	30·6 lb/in.³				55·1 lb/in.³			
	w_1	w_2	M_{x1}	M_{x2}	w_1	w_2	M_{x1}	M_{x2}
Experimental			0·680	−11·50			1·09	−12·50
Kantorovich	45	27	−0·440	−11·90	24	13	0·34	−13·16
Finite element 6 × 6	39	21	0·37	−11·57	24	12	0·91	−12·14
Multiplier	10^{-3} in.		lb in./in.		10^{-3} in.		lb in./in.	

(d) *Square tank on springs* (*Fig.* 8)

A square tank on springs and under hydrostatic loading is analysed for four different spring constants, the deflections and moments are compared with those obtained by Long [12], and, as can be seen from Table 4, the agreement is very good.

No difficulty will be encountered if a tank with a roof or a rectangular tank is used.

7. CONCLUSIONS

As seen from the examples quoted, the finite element procedure leads to very accurate results and presents no difficulties whatever kind of elastic foundation behaviour is assumed. Clearly this being the case, Winkler type spring approximation introduced to avoid mathematical difficulties need no longer be used where continuous foundations are presented.

It would have been observed that no special treatment of holes, corners or other irregularities in the plate is necessary and that therefore practical cases such as variable thickness foundation rafts, etc., are now capable of rapid solution.

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Résumé—Dans cette étude, les problèmes de plaques et de réservoirs (isotropes ou orthotropes), reposant soit sur un continuum élastique semi-infini soit sur ressorts individuels (du type que l'on appelle Winkler), sont résolus par la méthode à élement fini. Des coins rentrants, des murs rigides sur plaques, des moments concentrés dûs à la flexion de colonnes, etc., comportent peu de difficultés de calcul dans la méthode avancée.

Абстракт—В настоящей работе проблема плит или баков (изотропных или ортотропных) покоющихся на полу-бесконечном эластичном континууме или на отдельных пружинах (так называемые "типа Винклер") разрешена методом конечного элемента. Входящие углы, жесткие стенки плит, концентрированные моменты обусловленные изгибанием колонок, и т.п.—все это влечет лишь незначительные вычислительные затруднения при применении данного метода.